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On best rational approximation of analytic functions

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Abstract

This paper contains some theorems related to the best approximation $\rho_n(f;E)$ to a function *f* in the uniform metric on a compact set $E \subset \overline{\mathbf{C}}$ by rational functions of degree at most *n*. We obtain results characterizing the relationship between $\rho_n(f;K)$ and $\rho_n(f;E)$ in the case when complements of compact sets *K* and *E* are connected, *K* is a subset of the interior Ω of *E*, and *f* is analytic in Ω and continuous on *E*.

Keywords: Rational approximation; Degree of rational approximation; Singular number; Hankel operator; Meromorphic approximation

1. Meromorphic approximation and Hankel operators

1.1. Notation

Let $G \subset \mathbf{C}$ be a bounded domain and let Γ be the boundary of G. We assume that Γ consists of a finite number of closed analytic Jordan curves. Denote by $L_p(\Gamma)$, $1 \leqslant p < \infty$, the Lebesgue space of measurable functions φ on Γ with the norm given by

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the formula

$$||\varphi||_p = \left(\int_\Gamma |\varphi(\xi)|^p |d\xi| \right)^{1/p}.$$

We use the following notation for the inner product in the Hilbert space $L_2(\Gamma)$:

$$(\varphi, \psi) = \int_\Gamma \varphi(\xi)\overline{\psi}(\xi)|d\xi|, \quad \varphi, \psi \in L_2(\Gamma).$$

Let $L_\infty(\Gamma)$ be the space of essentially bounded on Γ functions with the norm

$$||\varphi||_\infty = \operatorname{ess\,sup}_\Gamma |\varphi(\xi)|, \quad \varphi \in L_\infty(\Gamma).$$

Denote by $E_p(G)$, $1 \leqslant p \leqslant \infty$, the Smirnov class of analytic functions on G. For $1 \leqslant p < \infty$ the class $E_p(G)$ consists of the functions φ for which there is a sequence of domains G_k with rectifiable boundaries having the following properties:

$$G_{k+1} \subset G_k, \quad \overline{G}_k \subset G, \quad \bigcup_k G_k = G$$

and

$$\sup_k \int_{\partial G_k} |\varphi(\xi)|^p |\,d\xi| < \infty.$$

$E_\infty(G)$ is the class of bounded analytic functions on G. The condition

$$\int_\Gamma \frac{\varphi(\xi)\,d\xi}{\xi - z} = 0 \quad \text{for all} \quad z \in \overline{\mathbf{C}} \setminus \overline{G}$$

is necessary and sufficient for a function $\varphi \in L_1(\Gamma)$ to be the boundary value of a function in the Smirnov class $E_1(G)$ (see [4,9] for more details about the classes $E_p(G)$).

Let σ be a positive Borel measure with support $\operatorname{supp} \sigma = F \subset G$. Let $L_q(\sigma, F)$, $1 \leqslant q < \infty$, be the Lebesgue space of measurable functions φ on F with the norm

$$||\varphi||_{q,\sigma} = \left(\int_F |\varphi(\xi)|^q \,d\sigma(\xi) \right)^{1/q}.$$

Denote by $J : E_2(G) \to L_2(\sigma, F)$ the *embedding* operator. The operator J is given by restricting an element $\varphi \in E_2(G)$ to F: $J\varphi = \varphi_{|F}$. It is not hard to see that J is a compact operator.

1.2. Auxiliary results from the theory of Hankel operators

Consider a function f continuous on Γ. We define the *Hankel operator* $A_f = A_{f,G} : E_2(G) \to E_2^\perp(G) = L_2(\Gamma) \ominus E_2(G)$ by

$$A_f(\varphi) = \mathbf{P}_-(\varphi f), \quad \varphi \in E_2(G),$$

where $\mathbf{P}_-$ is the orthogonal projection from $L_2(\Gamma)$ onto $E_2^\perp(G)$. The function f is called a symbol of the Hankel operator A_f. We remark that A_f is a compact operator. Let $\{s_n(A_f)\}$, $n = 0, 1, 2, \ldots,$ be the sequence of singular numbers of the operator A_f (the sequence of eigenvalues of the operator $(A_f^* A_f)^{1/2}$, where $A_f^* : E_2^{\perp}(G) \to E_2(G)$ is the adjoint of A_f). We assume that $s_0(A_f) \geqslant s_1(A_f) \geqslant \cdots \geqslant s_n(A_f) \geqslant \cdots$.

Let $\mathcal{M}_n(G)$ be the following class of meromorphic in G functions with at most n poles (counted with multiplicities):

$$\mathcal{M}_n(G) = \{h : h = p/q,\ p \in E_\infty(G),\ \deg q \leqslant n,\ q \not\equiv 0\}.$$

Let $\Delta_n(f; G)$ be the least deviation of f in the space $L_\infty(\Gamma)$ from the class $\mathcal{M}_n(G)$:

$$\Delta_n(f; G) = \inf_{h \in \mathcal{M}_n(G)} ||f - h||_\infty.$$

The AAK theorem (see [1]) asserts that for $G = \{z : |z| < 1\}$ and $f \in C(\Gamma)$, we have

$$s_n(A_f) = \Delta_n(f; G), \quad n = 0, 1, 2, \ldots .$$

In the case when G is a bounded domain and Γ consists of N closed analytic Jordan curves the following generalization of the AAK theorem was proved by the author (see [5]):

Let f be continuous on Γ. Then

$$s_n(A_f) \leqslant \Delta_n(f; G), \quad n = 0, 1, 2, \ldots, \tag{1}$$

and,

$$\Delta_{n+N-1}(f; G) \leqslant s_n(A_f), \quad \textit{for} \quad n \geqslant N - 1. \tag{2}$$

There exist (see [6]) orthonormal systems $\{q_n\},\ \{\alpha_n\},\ n = 0, 1, 2, \ldots,$ of eigenfunctions of the operator $(A_f^* A_f)^{1/2}$ corresponding to the sequence of singular numbers $\{s_n(A_f)\},\ n = 0, 1, 2, \ldots,$ such that

$$(q_n f - p_n)(\xi)\, d\xi = s_n(A_f)\overline{\alpha_n(\xi)}|\, d\xi| \quad \text{a.e. on} \quad \Gamma,$$

$$(\alpha_n f - \beta_n)(\xi)\, d\xi = s_n(A_f)\overline{q_n(\xi)}|\, d\xi| \quad \text{a.e. on} \quad \Gamma,$$

where $p_n, \beta_n \in E_2(G)$. Clearly,

$$\int_\Gamma (q_i \alpha_j)(\xi) f(\xi)\, d\xi = s_i(A_f)\delta_{i,j}, \quad i, j = 0, 1, 2, \ldots, \tag{3}$$

where $\delta_{i,j}$ is the Kronecker symbol.

We will need the following theorem (see [7]):

Theorem. *Let G be a bounded domain whose boundary consists of a finite number of closed analytic Jordan curves. Let f be continuous on Γ and let $\varphi_0, \ldots, \varphi_n \in E_2(G)$ and $\psi_0, \ldots, \psi_n \in E_2(G)$. Then the following estimate of the absolute value of a Hadamard-type determinant of order $n+1$ is valid*:

$$\left| \left| \int_\Gamma (\varphi_i \psi_j f)(\xi)\, d\xi \right|_{i,j=0}^{n} \right| \leqslant \prod_{k=0}^{n} s_k(A_f) \left(|(\varphi_i, \varphi_j)|_{i,j=0}^{n} \right)^{1/2} \left(|(\psi_i, \psi_j)|_{i,j=0}^{n} \right)^{1/2} \tag{4}$$

(*with the Gram determinants of order $n+1$ on the right*).

1.3. Estimates of errors in best meromorphic approximation

Let G be a bounded domain with boundary Γ consisting of N closed analytic Jordan curves. Consider a function f continuous on Γ. We assume that f can be extended analytically on $G \setminus F$, where F is a compact subset of G. Let G_1, $\overline{G}_1 \subset G$, be a domain bounded by a finite number of closed analytic Jordan curves which contains the compact set F. Denote by Γ_1 the boundary of G_1.We assume that Γ and Γ_1 are positively oriented with respect to G and G_1, respectively. Let $J : E_2(G) \to L_2(|\,dt|, \Gamma_1)$ be the corresponding embedding operator.

Theorem 1. *We have*

$$\prod_{k=0}^{n} s_k(A_{f,G}) \leqslant \prod_{k=0}^{n} s_k(A_{f,G_1}) \prod_{k=0}^{n} s_k^2(J).$$

We single out a result that follows directly from Theorem 1 (see (1) and (2)).

Corollary 2. *Let $N \geqslant 2$ and $n \geqslant N-1$. We have*

$$\prod_{k=0}^{N-2} s_k(A_{f,G}) \prod_{k=N-1}^{n} \Delta_{k+N-1}(f; G) \leqslant \prod_{k=0}^{n} \Delta_k(f; G_1) \prod_{k=0}^{n} s_k^2(J).$$

In the case when G is a simply connected domain we obtain the following:

Corollary 3. *Let G be a simply connected domain. Then*

$$\prod_{k=0}^{n} \Delta_k(f; G) \leqslant \prod_{k=0}^{n} \Delta_k(f; G_1) \prod_{k=0}^{n} s_k^2(J).$$

Proof of Theorem 1. Let $\{q_n\}, \{\alpha_n\}, n = 0, 1, 2, \ldots,$ be the orthonormal systems of eigenfunctions of the operator $(A^*_{f,G} A_{f,G})^{1/2}$ corresponding to the sequence of singular numbers $\{s_n(A_{f,G})\}, n = 0, 1, 2, \ldots,$ and satisfying the following equations (see (3)):

$$\int_\Gamma (q_i \alpha_j)(\xi) f(\xi)\, d\xi = s_i(A_{f,G})\delta_{i,j}, \quad i, j = 0, 1, 2, \ldots . \tag{5}$$

It follows immediately from (5) that the product of singular numbers $s_0(A_{f,G}) s_1(A_{f,G}) \ldots s_n(A_{f,G})$ can be written as a determinant of order $n+1$:

$$\prod_{k=0}^{n} s_k(A_{f,G}) = \left| \int_\Gamma (q_i \alpha_j)(\xi) f(\xi)\, d\xi \right|_{i,j=0}^{n}.$$

Since the functions $q_i, \alpha_j, i, j = 0, 1, 2, \ldots$, belong to $E_2(G)$ and f is analytic on $G \setminus F$, the formula

$$\prod_{k=0}^{n} s_k(A_{f,G}) = \left| \int_{\Gamma_1} (q_i \alpha_j)(t) f(t)\, dt \right|_{i,j=0}^{n} \tag{6}$$

can be written for the product of singular numbers.

Let $A_{f,G_1} : E_2(G_1) \to E_2^{\perp}(G_1)$ be the Hankel operator constructed from $f(t), t \in \Gamma_1$. Denote by $(q_i, q_j)_{2,|dt|}$ and (q_i, q_j) the inner products of q_i and q_j in the spaces $L_2(|dt|, \Gamma_1)$ and $L_2(\Gamma)$, respectively. From (6), by (4), we obtain that

$$\prod_{k=0}^{n} s_k(A_{f,G}) \leqslant \prod_{k=0}^{n} s_n(A_{f,G_1}) \left(\left|(q_i, q_j)_{2,|dt|}\right|_{i,j=0}^{n} \right)^{1/2} \left(\left|(\alpha_i, \alpha_j)_{2,|dt|}\right|_{i,j=0}^{n} \right)^{1/2}.$$

By the Weyl–Horn theorem (see, for example, [2, Lemma 3.1]),

$$\begin{aligned} \left|(q_i, q_j)_{2,|dt|}\right|_{i,j=0}^{n} &= \left|(Jq_i, Jq_j)_{2,|dt|}\right|_{i,j=0}^{n} \\ &\leqslant \prod_{k=0}^{n} s_k^2(J) \left|(q_i, q_j)\right|_{i,j=0}^{n} \end{aligned}$$

and

$$\begin{aligned} \left|(\alpha_i, \alpha_j)_{2,|dt|}\right|_{i,j=0}^{n} &= \left|(J\alpha_i, J\alpha_j)_{2,|dt|}\right|_{i,j=0}^{n} \\ &\leqslant \prod_{k=0}^{n} s_k^2(J) \left|(\alpha_i, \alpha_j)\right|_{i,j=0}^{n}. \end{aligned}$$

Taking into account now that $(\alpha_i, \alpha_j) = (q_i, q_j) = \delta_{i,j}$, we get

$$\begin{aligned} \prod_{k=0}^{n} s_k(A_{f,G}) &\leqslant \prod_{k=0}^{n} s_k(A_{f,G_1}) \prod_{k=0}^{n} s_k^2(J) \cdot \left(\left|(q_i, q_j)\right|_{i,j=0}^{n} \right)^{1/2} \left(\left|(\alpha_i, \alpha_j)\right|_{i,j=0}^{n} \right)^{1/2} \\ &\leqslant \prod_{k=0}^{n} s_k(A_f; G_1) \prod_{k=0}^{n} s_k^2(J). \quad \square \end{aligned}$$

2. Rational approximation

2.1. Estimates of errors in best rational approximation

Let E be an arbitrary compact set in the extended complex plane $\overline{\mathbf{C}}$. Consider a function f continuous on E. For any nonnegative integer n denote by $\rho_n(f; E)$ the best rational approximation of f in the uniform metric on E by rational function of order at most n. In other words,

$$\rho_n(f; E) = \inf_{r \in \mathcal{R}_n} ||f - r||_E,$$

where $||\cdot||_E$ is the supremum norm on E and the infimum is taken in the class of all rational functions of order at most n:

$$\mathcal{R}_n = \{r : r = p/q,\ \deg p \leqslant n,\ \deg q \leqslant n,\ q \not\equiv 0\}.$$

If f is analytic on $\bar{\mathbf{C}} \setminus F$, where F is a compact set in the extended complex plane $\overline{\mathbf{C}}$ such that $F \cap E = \emptyset$, then (see [6])

$$\limsup_{n\to\infty} \rho_n(f; E)^{1/n} \leqslant 1/\rho$$

and

$$\limsup_{n\to\infty} \left(\prod_{k=0}^{n} \rho_k(f; E) \right)^{1/n^2} \leqslant 1/\rho, \tag{7}$$

where $\rho = \exp(1/C(E, F))$ and $C(E, F)$ denotes the condenser capacity associated with the condenser (E, F) (see, for example, [3]).

Let $E \subset \overline{\mathbf{C}}$ be a compact set with connected complement U, $U \neq \emptyset$. We assume that the interior Ω of E is not empty. Denote by ∂E the boundary of E. Let f be a function analytic in Ω and continuous on E. We assume that f is not a rational function. It follows easily from this that $\rho_n(f; E) \neq 0$ for all $n = 0, 1, 2, \ldots$.

Let $K \subset \overline{\mathbf{C}}$ be a compact set and let K belong to the interior Ω of E. We assume that the complement G of K is connected.

Theorem 4. *We have*

$$\limsup_{n\to\infty} \left(\prod_{k=0}^{n} \rho_k(f; K) \Big/ \prod_{k=0}^{n} \rho_k(f; E) \right)^{1/n^2} \leqslant \exp(-1/C(\partial E, K)). \tag{8}$$

As a consequence of Theorem 4 we obtain the following result characterizing the asymptotics behavior of $\rho_n(f; K)/\rho_n(f; E)$ as $n \to \infty$.

Corollary 5. *The following inequality is valid*:

$$\liminf_{n\to\infty} \left(\frac{\rho_n(f; K)}{\rho_n(f; E)} \right)^{1/n} \leqslant \exp(-2/C(\partial E, K)).$$

Proof of Theorem 4. We first assume that K and E are bounded by finitely many disjoint closed analytic Jordan curves. Since quantities $\rho_n(f; K)$, $\rho_n(f; E)$, $n = 0, 1, \ldots,$ and the condenser capacity $C(\partial E, K)$ are invariant under linear fractional transformations of the extended complex plane $\overline{\mathbf{C}}$ we confine ourselves to the case when the complement of K (the domain G) is bounded.

It is not hard to see that $\Omega = \overline{\mathbf{C}} \setminus \overline{U}$. Moreover, since U is connected, $\overline{U}$ is a continuum (a closed connected set with at least two points). Hence, Ω consists of a finite number of simply connected domains bounded by closed analytic Jordan curves.

Let $w(z)$ be the solution of the Dirichlet problem constructed in the open set $\Omega \setminus K$ with respect to boundary data equal 1 on ∂K and 0 on $\partial\Omega$. It will be assumed that $w(z)$ is extended by continuity to $\overline{\mathbf{C}} : w(z) = 1$ for $z \in K$, and $w(z) = 0$ for $z \in \overline{U}$. For arbitrary ε, with $0 < \varepsilon < 1$, let $G(\varepsilon) = \{z : w(z) < \varepsilon\}$ and $\gamma(\varepsilon) = \{z : w(z) = \varepsilon\}$.

Using the maximum principle for harmonic functions we can conclude that every connected component of the open set $G(\varepsilon)$, $0 < \varepsilon < 1$, contains at least one point of $\overline{U}$. Since $\overline{U}$ is a continuum, it follows from this that $G(\varepsilon)$ is a domain. We assume that $\gamma(\varepsilon)$, $0 < \varepsilon < 1$, is positively oriented with respect to the domain $G(\varepsilon)$. We distinguish components Ω_i of Ω such that $\Omega_i \cap K \neq \emptyset$. Let $\Omega' = \bigcup_i \Omega_i$. Note that $\Omega' \subset \Omega$. By the properties of the condenser capacity (see, for example, [3,8]),

$$C(\partial\Omega', \partial K) = C(\partial\Omega, \partial K) = C(\partial\Omega, K).$$

Since $\partial\Omega = \partial E$, we obtain from this that

$$C(\partial\Omega', \partial K) = C(\partial E, K).$$

We have (see, for example, [3,8])

$$\lim_{\varepsilon\to 0, \varepsilon_1 \to 1} C(\gamma(\varepsilon), \gamma(\varepsilon_1)) = C(\partial\Omega', \partial K).$$

So,

$$\lim_{\varepsilon\to 0, \varepsilon_1 \to 1} C(\gamma(\varepsilon), \gamma(\varepsilon_1)) = C(\partial E, K). \tag{9}$$

Fix $0 < \varepsilon < \varepsilon_1 < 1$. We assume that ε and ε_1 are choosen close enough to 0 and 1, respectively, such that $\gamma(\varepsilon)$ and $\gamma(\varepsilon_1)$ consist of disjoint closed analytic Jordan curves. Let $\gamma(\varepsilon_1)$ consist of N closed analytic Jordan curves. Denote by $J : E_2(G(\varepsilon_1)) \to L_2(|dt|, \gamma(\varepsilon))$ the corresponding embedding operator. Since f is analytic in $\overline{\mathbf{C}} \setminus \overline{U}$, and since $\overline{U} \subset G(\varepsilon) \subset \overline{G(\varepsilon)} \subset G(\varepsilon_1)$, it follows from Corollary 2, that for $n \geqslant N - 1$

$$\prod_{k=N-1}^{n} \Delta_{k+N-1}(f; G(\varepsilon_1)) \leqslant C \prod_{k=0}^{n} \Delta_k(f; G(\varepsilon)) \prod_{k=0}^{n} s_k^2(J), \tag{10}$$

where C is a positive quantity not depending on n. Here and in what follows denote by $C, C_1, \ldots,$ positive quantities not depending on n.

Let us estimate

$$\prod_{k=0}^{n} \Delta_k(f; G(\varepsilon)).$$

It follows from the definitions of $\Delta_k(f; G(\varepsilon))$ and $\rho_k(f; \gamma(\varepsilon))$ that

$$\Delta_k(f; G(\varepsilon)) \leqslant \rho_k(f; \gamma(\varepsilon)).$$

Since $\gamma(\varepsilon) \subseteq E$, we can write

$$\rho_k(f; \gamma(\varepsilon)) \leqslant \rho_k(f; E).$$

So,

$$\prod_{k=0}^{n} \Delta_k(f; G(\varepsilon)) \leqslant \prod_{k=0}^{n} \rho_k(f; E),$$

and, by (10),

$$\prod_{k=N-1}^{n} \Delta_{k+N-1}(f; G(\varepsilon_1)) \leqslant C \prod_{k=0}^{n} \rho_k(f; E) \prod_{k=0}^{n} s_k^2(J). \tag{11}$$

Fix a nonnegative integer k. For an arbitrary rational function $r \in \mathcal{R}_k$ with poles outside $\gamma(\varepsilon)$ and any function $\varphi \in E_\infty(G(\varepsilon_1))$ we have by the Cauchy formula

$$(r' - f)(z) = \frac{1}{2\pi i} \int_{\gamma(\varepsilon_1)} \frac{(f - r - \varphi)(\xi)\, d\xi}{\xi - z}, \quad z \in K, \tag{12}$$

where r' is the sum of the principal parts of r corresponding to poles of r lying in $G(\varepsilon_1)$. We remark that $r' \in \mathcal{R}_k$. Estimating the integral in (12), we get

$$\rho_k(f; K) \leqslant ||f - r'||_K \leqslant C_1 ||f - r - \varphi||_\infty.$$

Since r is an arbitrary function in $\mathcal{R}_k$ with poles outside $\gamma(\varepsilon_1)$ and φ is an arbitrary function in $E_\infty(G(\varepsilon_1))$,

$$\rho_k(f; K) \leqslant C_1 \Delta_k(f; G(\varepsilon_1)).$$

From this, by (11), we can write

$$\prod_{k=0}^{n} \rho_k(f; K) \leqslant C_2^n \prod_{k=0}^{n} \rho_k(f; E) \prod_{k=0}^{n} s_k^2(J). \tag{13}$$

Using the result of Zaharjuta and Skiba (see [10]),

$$\lim_{n\to\infty} s_n^{1/n}(J) = \exp(-1, C(\gamma(\varepsilon), \gamma(\varepsilon_1)))$$

from (13) we get

$$\limsup_{n\to\infty} \left(\prod_{k=0}^{n} \rho_k(f; K) \Bigg/ \prod_{k=0}^{n} \rho_k(f; E) \right)^{1/n^2} \leqslant \exp(-1/C(\gamma(\varepsilon), \gamma(\varepsilon_1))).$$

Letting $\varepsilon \to 0$ and $\varepsilon_1 \to 1$, we obtain (see (9)) that

$$\limsup_{n\to\infty} \left(\prod_{k=0}^{n} \rho_k(f; K) \Bigg/ \prod_{k=0}^{n} \rho_k(f; E) \right)^{1/n^2} \leqslant \exp(-1/C(\partial E, K)). \tag{14}$$

We now get rid of the condition that K and E are bounded by finitely many closed analytic Jordan curves. Consider the general case when K and E are arbitrary compact sets satisfying the conditions:

(a) $K \subset \Omega$;
(b) $G = \overline{\mathbf{C}} \setminus K$ and $U = \overline{\mathbf{C}} \setminus E$ are connected, $U \neq \emptyset$.

Since U is the complement of E, $\partial E = \partial U$, and $E = \partial E \cup \Omega$, it follows that $\overline{U} = U \cup \partial E$ and Ω coincides with the complement of a closed domain $\overline{U}$. Using now the fact that U is the complement of E, and Ω is the complement of $\overline{U}$, we can write

$$\partial\Omega = \partial\overline{U} \subseteq \partial U = \partial E. \tag{15}$$

Since $\overline{U}$ is a continuum, Ω consists of an at most countable number of simply connected domains. We distinguish components Ω_i of Ω such that $\Omega_i \cap K \neq \emptyset$. Since Ω is an open cover of the compact set K, it follows that there is only a finite number of such components Ω_i. Let $\Omega' = \bigcup_i \Omega_i$. We remark that $\Omega' \subset \Omega$. By the properties of the condenser capacity

$$C(\partial\Omega', K) = C(\partial\Omega, K).$$

So, by (15), we can write

$$C(\partial\Omega', K) \leqslant C(\partial E, K). \tag{16}$$

Let $B = \overline{\mathbf{C}} \setminus \Omega'$. Since $\overline{U}$ is a continuum, we can conclude that B is a continuum. Moreover, since $K \subset \Omega'$, $B \cap K = \emptyset$. We construct a sequence of compacts $\{K_m\}$ and $\{B_m\}$, $m = 1, 2, \ldots,$ bounded by finitely many closed analytic Jordan curves, that tends monotonically to K and B, respectively:

$$K \subset K_m \subset K_{m-1}, \quad \bigcap_{m=1}^{\infty} K_m = K,$$

$$B \subset B_m \subset B_{m-1}, \quad \bigcap_{m=1}^{\infty} B_m = B.$$

We assume that for all m, B_m is a continuum, the complement of K_m is connected, and $B_m \cap K_m = \emptyset$.

Fix a positive integer m. Let V_m be the closure of the complement of B_m in the extended complex plane $\overline{\mathbf{C}}$. It is easy to see that $V_m \subset \Omega' \subset E$. Since B_m is a continuum, the complement of V_m is connected. Using the relations $K \subset K_m$ and $V_m \subset E$, for all nonnegative integers n and m we can write

$$\rho_n(f; K) \leqslant \rho_n(f; K_m) \tag{17}$$

and

$$\rho_n(f; V_m) \leqslant \rho_n(f; E). \tag{18}$$

Since K_m and V_m are bounded by finitely many closed analytic Jordan curves, with the help of estimate (14) we get

$$\limsup_{n\to\infty}\left(\prod_{k=0}^{n}\rho_k(f;K_m)\Big/\prod_{k=0}^{n}\rho_k(f;V_m)\right)^{1/n^2}\leqslant \exp(-1/C(\partial V_m,K_m)).$$

This implies (see (17) and (18))

$$\limsup_{n\to\infty}\left(\prod_{k=0}^{n}\rho_k(f;K)\Big/\prod_{k=0}^{n}\rho_k(f;E)\right)^{1/n^2}\leqslant \exp(-1/C(\partial V_m,K_m)). \tag{19}$$

By properties of the condenser of the capacity we have

$$\lim_{m\to\infty} C(\partial V_m, K_m) = C(\partial\Omega', K).$$

So, we can pass to the limit on the right-hand side of (19), obtaining

$$\limsup_{n\to\infty}\left(\prod_{k=0}^{n}\rho_k(f;K)\Big/\prod_{k=0}^{n}\rho_k(f;E)\right)^{1/n}\leqslant \exp(-1/C(\partial\Omega',K)).$$

Using now (16), we get (8). □

Let a function f be analytic in an open set D and let $E \subset D$ be a compact set with connected complement. We assume that D consists of a finite number of domains D_i, $i = 1,\dots,\chi$, and $D_i \cap E \neq \emptyset$ for all i. Denote by F the complement of D in the extended complex plane $\overline{\mathbf{C}}$. It is assumed that F is a continuum. It follows from this that the logarithmic capacity cap (F) (see [3,8]) of F is positive and F is a regular compact set in the sense of potential theory.

Let $\rho = \exp(1/C(E,F))$, where $C(E,F)$ is the condenser capacity associated with the condenser (E,F). We assume that the logarithmic capacity cap(E) is positive. From this and the fact that cap$(F) > 0$ we can conclude that (see [3,8]) that $C(E,F) > 0$.

Denote by $w(z)$ the solution of the generalized Dirichlet problem with the boundary function equal to 1 on ∂F and to 0 on ∂E. For each $i = 1,\dots,\chi$, the function $w(z)$ is harmonic in the domain $D_i \setminus E$. It is assumed that the compact set E is regular. Since E and F are regular compacts, $w(z)$ is continuous on $\overline{D\setminus E}$; $w(z) = 1, z \in \partial D = \partial F$, and $w(z) = 0, z \in \partial E$. It will be assumed that $w(z)$ is extended by continuity to $\overline{\mathbf{C}}$: $w(z) = 1$ for $z \in F$ and $w(z) = 0$ for $z \in E$. For arbitrary r, with $1 < r < \rho$, let $E(r) = \{z : w(z) \leqslant \ln r/\ln\rho\}$ and $\gamma(r) = \{z : w(z) = \ln r/\ln\rho\}$. We remark that, by properties of the condenser capacity (see, for example, [3,8]),

$$C(E,\gamma(r)) = \frac{\ln\rho}{\ln r}C(E,F)$$

and

$$\exp(1/C(E,\gamma(r))) = r. \tag{20}$$

Using (7), it is easy to obtain an upper estimate for $\liminf_{n\to\infty} \rho_n(f;E)^{1/n}$:

$$\liminf_{n\to\infty} \rho_n(f;E)^{1/n} \leqslant \frac{1}{\rho^2}.$$

We conclude this section with the result related to functions *f* having the following asymptotics of the errors in the best rational approximation:

$$\lim_{n\to\infty} \rho_n(f;E)^{1/n} = \frac{1}{\rho^2}.$$

Theorem 6. *Let*

$$\lim_{n\to\infty} \rho_n(f;E)^{1/n} = \frac{1}{\rho^2},$$

where $\rho = \exp(1/C(E,F))$. *Then for any* $1 < r < \rho$,

$$\lim_{n\to\infty} \rho_n(f;E(r))^{1/n} = \exp(-2/C(E(r),F)) = \left(\frac{r}{\rho}\right)^2. \tag{21}$$

Proof. Since

$$\lim_{n\to\infty} \rho_n(f;E)^{1/n} = \frac{1}{\rho^2},$$

we can write

$$\lim_{n\to\infty} \left(\prod_{k=0}^{n} \rho_k(f;E)\right)^{1/n^2} = \frac{1}{\rho}. \tag{22}$$

Using (8) and (20), we get

$$\limsup_{n\to\infty} \left(\prod_{k=0}^{n} \rho_k(f;E) \Big/ \prod_{k=0}^{n} \rho_k(f;E(r))\right)^{1/n^2} \leqslant \exp(-1/C(E,\gamma(r)) = \frac{1}{r}. \tag{23}$$

From this, by (22), we obtain

$$\liminf_{n\to\infty} \left(\prod_{k=0}^{n} \rho_k(f;E(r))\right)^{1/n^2} \geqslant \frac{r}{\rho}. \tag{24}$$

Since

$$C(E(r),F) = C(E,F)/(1-\ln r/\ln\rho)$$

and

$$\exp((1/C(E(r),F)) = \frac{\rho}{r},$$

we have (see (7)),

$$\limsup_{n\to\infty}\left(\prod_{k=0}^{n}\rho_k(f;E(r))\right)^{1/n^2}\leqslant \exp(-1/C(E(r),F))=\frac{r}{\rho}.$$

So, by (24),

$$\lim_{n\to\infty}\left(\prod_{k=0}^{n}\rho_k(f;E(r))\right)^{1/n^2}=\frac{r}{\rho}. \tag{25}$$

Fix an arbitrary $0<\theta<1$. Choose a sequence of integers $\{k_n\}, n=1,2,3,\ldots,$ such that $0\leqslant k_n\leqslant n, \lim_{n\to\infty} k_n/n=\theta$. Since the sequence $\{\rho_n(f;E(r))\}, n=1,2,\ldots$ is nonincreasing,

$$\left(\prod_{k=0}^{k_n}\rho_k(f;E(r))\right)\rho_n^{n-k_n}(f;E(r))\leqslant\prod_{k=0}^{n}\rho_k(f;E(r)). \tag{26}$$

From (26), on account of (25), we obtain that

$$\limsup_{n\to\infty}\rho_n(f;E(r))^{1/n}\leqslant\left(\frac{r}{\rho}\right)^{1+\theta}.$$

Letting $\theta\to 1$, we get

$$\limsup_{n\to\infty}\rho_n(f;E(r))^{1/n}\leqslant\left(\frac{r}{\rho}\right)^{2}. \tag{27}$$

Using now the inequality

$$\left(\prod_{k=0}^{k_n}\rho_k(f;E(r))\right)\leqslant\rho_n^{k_n-n}(f;E(r))\prod_{k=0}^{n}\rho_k(f;E(r)),$$

where $k_n\geqslant n$, and the same arguments as above it is not hard to prove the following:

$$\liminf_{n\to\infty}\rho_n(f;E(r))^{1/n}\geqslant\left(\frac{r}{\rho}\right)^{2},$$

which with help of (27) implies the desired equality (21). □

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